A Perturbation Solution of Helicopter Rotor Flapping Stability

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Theme

A SOLUTION for the flapping stability of a helicopter blade, including the influence of a periodic coefficients due to the rotor forward velocity, is obtained using the techniques of perturbation theory.

Contents

The dynamics of a helicopter rotor in forward flight are described by a set of linear, ordinary differential equations with periodic coefficients. Sophisticated and well-known techniques are available for the analysis of constant coefficient equations, including stability determination and control system design. Even the determination of the stability of periodic coefficient equations requires, however, considerable numerical calculation (specifically, for one frequently used technique, numerical integration of the equations of motion is required), and other topics are considerably more difficult than the corresponding constant coefficient system analysis. An alternate mathematical approach to periodic coefficient equations is the use of small perturbation techniques. The base paper considers the application of these techniques to the calculation of the flapping stability of a single blade of a helicopter rotor.

The dynamics of the flapping motion of a single blade of a helicopter rotor are governed by the following equation.

$$\ddot{\beta} + \nu^2\beta = \gamma[(M_{\dot{\beta}} - K_R M_\theta)\dot{\beta} + (M_\beta - K_p M_\theta)\beta] \tag{1}$$

β is the degree of freedom representing the perturbed motion of the blade out of the plane of the rotor rotation; for an articulated rotor it is just the angle of rotation about the flapping hinge. The time variable, nondimensionalized by the rotor speed, is the azimuth angle ψ. The rotating natural frequency of the flapping motion, divided by rotor rotational speed, is ν. The blade Lock number γ, representing the ratio of aerodynamic to inertial forces for the blade, is defined by $\gamma = \rho acR^4/I_b$ (ρ is the air density, a is the two-dimensional lift curve slope, c the blade chord, R the rotor radius, and I_b the flapping moment of inertia of the blade). The flap proportional feedback gain K_P is better known as $\tan\delta_3$; K_R is the flap rate feedback gain. A feedback law $\Delta\theta = -K_p\beta - K_R\dot{\beta}$ has been used, where $\Delta\theta$ is the blade pitch change due to flapping feedback control. The coefficients M_β, $M_{\dot{\beta}}$, and M_θ are the aerodynamic flapping moments on the blade; they are functions of ψ and μ, where μ is the rotor advance ratio, i.e., rotor forward velocity divided by tip speed. In hover, $\mu = 0$, the coefficients are simple constants, independent of ψ; but in forward flight, $\mu > 0$, the coefficients become periodic functions of ψ (due to the periodic variation of the velocity seen by the blade as it moves around the disk). In general then, Eq. (1) is a linear, ordinary differential equation, with periodic coefficients. Expressions for the coefficients are given in the base paper, and their derivation may be found in the literature.[1]

It is the periodic coefficients of Eq. (1) which require the use of more complicated mathematical techniques than are necessary for the analysis of the dynamics of systems with linear, constant coefficient equations. The stability of Eq. (1) has been studied[2] using Floquet theory and numerical techniques. The techniques of perturbation theory allow such equations to be examined entirely analytically, and so offer great potential for the study of rotor dynamics. The base paper applies the techniques of perturbation theory to obtain analytic solutions for the stability of Eq. (1). There are two parameters which are available for use as perturbation quantities: the advance ratio μ and the Lock number γ. There are then four cases to be considered: small and large μ, and small and large γ. Since small μ is characteristic of the operation of most helicopters, the small μ case is in many aspects the most useful case of the four; this synoptic summarizes the results of the small μ case. Results are presented for the roots or eigenvalues of the equation (there are two for this second order equation), since these determine the stability of the motion.

The perturbation technique known as the method of multiple time scales[3] is used to obtain the solution for small μ. The technique involves the expansion of the variable β and the time derivatives $d/d\psi$ and $d^2/d\psi^2$ as series in powers of μ; and the parameters ν and γ are also expanded as such series. Only the case with $K_R = 0$, no flap rate feedback, is presented here. The hover ($\mu = 0$) solution for the eigenvalues is obtained as

$$\lambda = -\gamma/16 + i[\nu^2 + (\gamma/8)K_P - (\gamma/16)^2]^{1/2} \tag{2}$$

and its conjugate. For most values of the parameter γ and ν, the roots to order μ^2 are

$$\lambda = -\gamma/16 + i\Big[\nu^2 + (1+\mu^2)(\gamma/8)K_P - (\gamma/16)^2\Big(1 - \mu^2(8/9)\frac{\nu^2 - (\gamma/8)K_P + 4K_P^2}{(\gamma/16)^2 - \nu^2 - (\gamma/8)K_P + 1/4}\Big)\Big]^{1/2} \tag{3}$$

and its conjugate. Equation (3) applies in particular when there are two real roots (the quantity in brackets negative), and so gives the criterion for simple divergence, including the effects of the periodic coefficients. The divergence boundary is given by $\lambda = 0$, from which there follows

$$(1+\mu^2)(\gamma/8)K_P = -\nu^2\Big(1 + \mu^2(16/9)\frac{(\gamma/16)^2 + \nu^2/2}{(\gamma/16)^2 + 1/4}\Big) \tag{4}$$

This is primarily a limit of flap proportional feedback, K_P; the boundary is crossed when K_P is sufficiently large negative (positive pitch/flap coupling). For larger negative values of K_P one of the real roots is positive, and the system is unstable.

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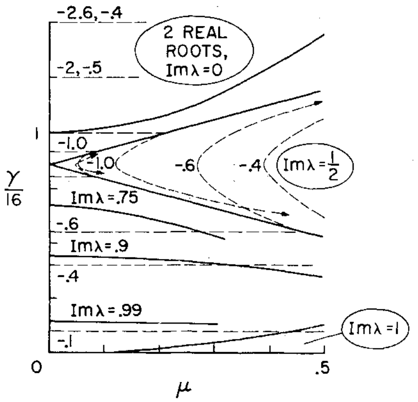

Fig. 1 Lines of constant $Im\lambda$ and $Re\lambda$, based on the small μ results (to order μ^2); $\nu = 1$, $K_P = 0$; —$Im\lambda$, - - - $Re\lambda$, circled values of $Im\lambda$ indicate areas in which $Im\lambda$ is constant.

It is the characteristic of systems with periodic coefficients that for certain values of the parameters there occurs a degradation of the stability, perhaps even instability. This typically occurs where the basic eigenvalue has a frequency (imaginary part) corresponding to a multiple of one-half the fundamental frequency of the equation coefficients. When the frequency of the root is near ½/rev, Eq. (3) no longer is valid, and the following result must be used instead. Let γ_0 be the value of γ for which the hover root [Eq. (2)] has frequency exactly ½/rev, and write $\Delta\gamma = \gamma - \gamma_0$. If $\Delta\gamma/16$ is order μ small, then the eigenvalue is given by

$$\lambda = -\gamma/16 + i/2$$
$$-i(\Delta\gamma/16)2(\gamma/16 - K_P)[1 - (\mu/\mu_{\text{corner}})^2]^{1/2} \qquad (5)$$

where

$$\mu_{\text{corner}} = (\Delta\gamma/16)2(\gamma/16 - K_P)/\{(\gamma/24)[1 + (\gamma/8 - 4K_P)^2]^{1/2}\}$$

[If $K_P = 0$ then $\mu_{\text{corner}} = \Delta\gamma/16(3/2\nu)$]. The subscript "corner" refers to the behavior of the μ root locus on the λ plane. If $\mu < \mu_{\text{corner}}$ then there is an order μ change in the frequency, the roots moving toward ½/rev, while the real part of the root is fixed at $-\gamma/16$. At $\mu = \mu_{\text{corner}}$ the roots reach $Im\lambda = $ ½/rev. For $\mu > \mu_{\text{corner}}$ the frequency remains fixed at ½/rev while there is an order μ change in the damping; $Re\lambda$ is increased for one root and decreased for the other. This behavior of the locus near $Im\lambda = $ ½/rev is typical of periodic systems. While there is a stability degradation if μ is large enough ($> \mu_{\text{corner}}$, which decreased as $\Delta\gamma$ decreased, i.e., as the hover root approaches ½/rev), the reduction in damping is order μ small. The hover damping, $Re\lambda = -\gamma/16$, is quite large for usual values of γ, and so stability is maintained for small μ, even with the influence of the periodic coefficients.

Similar behavior is exhibited when the hover root frequency is near 1/rev. Let $\Delta\gamma = \gamma - \gamma_0$, where now γ_0 is the value for which the hover root is exactly at 1/rev; then if $\Delta\gamma/16$ is order μ^2 small the roots are given by

$$\lambda = -\gamma/16 + i$$
$$-i(\Delta\gamma/16)(\gamma/16 - K_P)[(1 - \mu^2/\mu_1^2)(1 - \mu^2/\mu_2^2)]^{1/2} \qquad (6)$$

The quantities μ_1 and μ_2 are functions of γ, ν, and K_P (and are given in the base paper). For small μ there is only an order μ^2 change in the frequency, toward 1/rev, while there is no change in the real part from the hover value. For $\mu = \mu_1$ or μ_2 the roots reach $Im\lambda = 1$/rev. For still larger μ, the frequency is fixed at 1/rev while there is an increase of the real part of one root and a decrease of the other. The stability degradation is order μ^2 small, so again the system will remain stable for small μ and reasonable γ.

The eigenvalues depend primarily on γ and μ, so the above results may be presented as contours of constant $Re\lambda$ and constant $Im\lambda$ on the $\gamma - \mu$ plane. Such a plot is shown in Fig. 1 for $\nu = 1$ and $K_P = 0$. The regions in which $Im\lambda$ is fixed at ½/rev or 1/rev are the effect of the periodic coefficients. A horizontal line on the figure is a line of constant γ; so the variation of $Re\lambda$ and $Im\lambda$ as such a line is traversed gives the root locus for varying μ. Consider for example a horizontal line with $\gamma = 8$ ($\gamma/16 = 0.5$); as μ increases from zero, the line remains parallel to the $Re\lambda$ = constant lines so $Re\lambda$ remains fixed at the hover value. The $Im\lambda$ = ½ region comes closer to the horizontal line, which means that $Im\lambda$ moves toward ½/rev. At the corner μ the locus crossed into the $Im\lambda$ = ½ region; for higher μ then $Im\lambda$ remains fixed at ½/rev while for each point in the region there are two values of $Re\lambda$, one more and one less stable than the hover root.

A comparison with numerical solutions of Eq. (1) indicates that the small μ perturbation solution to order μ^2 is accurate to about $\mu = 0.5$; this solution then covers the range of interest for most helicopters. The base paper also presents the perturbation solutions for the large μ, small γ, and large γ cases. A more detailed summary[4] of the base paper presents a comparison with the numerical results of Peters and Hohenemser.[2]

Perturbation theory has provided a solution to the basic problem of helicopter rotor flapping stability. Further steps in the application of these techniques to helicopter dynamics should include shaft motion or other blade degrees of freedom in the analysis. The ultimate goal is to determine when the influence of the periodic coefficients is important and to provide methods to treat such cases, particularly more involved topics such as control system design.

References

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